JL Seminar

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1 Introduction

1.1. Our goal is to understand irred. adm. repn. of $\mathrm{GL}_n(F)$ for $n = 2$, F local narc.

1.1 Writing conventions

1.2. I will be using many shorthands, generally following a "syllabic abbreviation", i.e.

- ext. : extension. With first three letters for the type of extensions.
  - alg./sep. : algebraic/separable
- cplt./cpt./td.: complete/compact/totally disconnected.
- wrt./narc. : with respect to/ non-archimedean.

In general, the context (ctx) should make it clear what I'm talking about.

1.2 Notation

1.3. We let F be a fixed loc. field.

- $\mathcal{O}_F$ its ring of integers, $\mathcal{O}_F^\times$ its grp of units.
- $\mathfrak{p}$ max. ideal of $\mathcal{O}_F$ with ϖ a gen. of $\mathfrak{p}$.
- ψ a fixed ntriv. add. char. of F.

1.4. We follow the notation in [JL70] with minor modification. Let $G_F := \mathrm{GL}_2(F)$ we describe several *sbgps*

- $K_F := \mathrm{GL}_2(\mathcal{O}_F)$, is also a[1] max. cpt. open sbgrp.
- Z_F is center of G_F consisting of scalar matrices, hence iso. to $F^\times$.
- D_F be sbgrp. of matrices of the from $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $a \in F^\times, b \in F$.
- B_F is sbgrp. of matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, also known as *Borel subgrp*, $a, d \in F^\times, b \in F$.
- N_F is sbgrp. of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in F$. We thus have an identification

$$F \xrightarrow{\simeq} N_F, x \mapsto n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

- T_F is subgrp. of diagonal matrices. the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $a, d \in F^\times$.
- C_F is subgrp. of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $a \in F^\times$.

1.5. Mapping spaces. Let X be a space, $V \in \mathrm{Vect}_{\mathbb{C}}$.

- $\mathrm{Map}(X, V)$ is the *set* of V-valued fncs.
- $\mathrm{Map}^\infty(X, V)$ " loc. const. V-valued fncs.
- $\mathrm{Map}_c^\infty(X, V)$ " loc. const. cptly supported V-value fncs.

Remark 1.6. When $V = \mathbb{C}$, we often omit the V. The second and third type are also called *smooth* and *schwartz* functions respectively, denoted as $C^\infty(X, V)$ and $\mathcal{S}(X, V)$ in [JL70].

[1] is this *the*?

2 Overview

2.1. [PS83, 13] The method of constructing repns consists of three stages.

1. Use general methods to construct representations of D_F.
2. Then we "jump" to B_F an induce characters from B_F to G.
3. The last is to explore those repns that do not appear. (hardest).

2.2. Whittaker models come about at step 1. These correspond to induced representations from N_F.

2.1 Structure on subgroups

2.3. Structure of B_F.

- B_F is a solvable grp [2], whose normal abelian gp is U_F
- N_F and D_F and normal subgroup of B_F.
- We have the followin two decompositions for B_F

$$B_F = D_F \rtimes Z_F = N_F \rtimes A_F$$

2.4. Structure of D_F.

- $D_F = N_F \rtimes C_F$.
- The action of C_F on N_F is by conjugation of $F^\times$ on F^+, i.e.

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha\beta \\ 0 & 1 \end{pmatrix}$$

2.2 Kirillov model

2.5. Kirillov representation of D_F. It $V \subset \mathrm{Map}(F^\times, \mathbb{C})$, *complex valued functions*, on which D_F operates by

$$\pi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\xi(x) = \psi_F(bx)\xi(ax)$$

then π is a *Kirillov representation*. This also restircts to an action on $\mathrm{Map}^\infty(F^\times, \mathbb{C}), \mathrm{Map}_c^\infty(F^\times, \mathbb{C})$. We denote this repn. as

$$(\xi_\psi, \mathrm{Map}(F^\times))$$

Definition 2.6. A *Kirillov model* of (π, V), is an equiv. repn. of G_F on a subspace of $V' \subset \mathrm{Map}(F^\times)$ such that the canonical inclusion $D_F \hookrightarrow G_F$ identifies $\mathrm{Res}_{D_F}^{G_F} V'$ as a submodule of $(\xi_\psi, \mathrm{Map}(F^\times))$. Here

$$\mathrm{Res}_{D_F}^{G_F} : \mathrm{Rep}(G_F) \to \mathrm{Rep}(D_F)$$

is the restrict. functor (left adj. to induction).

Theorem 2.7. Let (π, V) be an admissible infinite dimensional representation of G_F. Then π has a unique Kirillov model.

[2] i.e. there is a subnormal series whose factors are abelian.

Proof. Step 0. (π, V) *is a Pre-Kirillov model: we can identify* V *as a subspace of* $\mathrm{Map}^\infty(F^\times, J_\psi V)$.

Step 1. Understanding this space.

Step 2. Understanding the action of G_F. □

2.8. A key input in *Step 2* is understanding the structure theory of G_F, it can be decomposed to three types of matrices.

- Diagonal.
- D_F.
- $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We will need a generalized version of Mellin transform.

2.9. We will end with showing that for an irred. adm. inf. dim. rep. (π, V):

1. $J_\psi V$ is one dimensional.
2. π admits a unique Kirillov model.
3. π admits a unique Whittaker model.

3 Uniqueness of models

3.1 Representations and functionals on Schwartz Space

[JL70, 2]

Definition 3.1. We define a representation (ξ_ψ, D_F) on the spaces $\mathrm{Map}(F, X)$ and $\mathrm{Map}(F^\times, X)$ by [3]

$$\left(\xi_\psi \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \phi\right)(y) = \psi(yx)\phi(ya) \tag{1}$$

This also induces action on $\mathrm{Map}^\infty(F, X), \mathrm{Map}_c^\infty(F, X)$ etc.

Lemma 3.2. [JL70, 2.13.3] Let ϕ be an element of $\mathcal{S}(F^\times)$. Then there exists

- A finite subset S of $F^\times$
- Complex numbers $\lambda_y \in S$ where
$$\sum \lambda_y = 0, \quad \sum \lambda_y \psi(y) = \phi(1)$$
- an element $\phi_0 \in \mathrm{Map}_c^\infty(F^\times)$.

such that
$$\phi = \sum_{y \in S} \lambda_y \phi_\psi(n_y)\phi_0$$

[3] Note that the action of a is on the right.

Proof. Step 1. Fourier transformation Extend ϕ to a function on F - this is still an element on $\mathcal{S}(F)$. Let ϕ' denote the Fourier transform of ϕ. [4]

Step 2. Discreteness Then the function

$$F \times F \to \mathbb{C}, \quad (y, x) \mapsto \phi'(-y)\psi(xy)$$

is loc. const. and cptly. sup.
Step 2. Evaluation □

Corollary 3.3. [JL70, 2.13.1] Let L be a linear functional on Schwartz space $\mathcal{S}(F^\times)$ satisfying

$$L(\xi_\psi(n_x)\phi) = \psi(x)L(\phi)$$

for all ϕ in $\mathcal{S}(F^\times)$ and all $x \in F$. Then there is a scalar λ such that

$$L(\phi) = \lambda\phi(1)$$

Proof. Step 0. A linear reduction. As open subgrps of top. groups are also closed, 3. of 5.3, char. fncs. 1_U, where U is an open sbgrp, lies in $\mathrm{Map}^\infty(F^\times)$ and in $\mathrm{Map}_c^\infty(F^\times) = \mathcal{S}(F^\times)$ if U is cpt.

Hence, given $\phi \in \mathcal{S}(F^\times)$, replacing subtracting by $\phi(1)1_U$, we have

$$L(\phi - \phi(1)1_U) = L(\phi) - \phi(1)L(1_U)$$

If we can prove *Step 1.* below, we have obtained the desired form with $\lambda L(1_U)$.

Step 1. Use the representation in 3.2 □

3.2 Uniqueness of Whittaker functional

Definition 3.4. (π, V) be as ctx. A *Whittaker functional* on V is a linear map $L : V \to \mathbb{C}$, st.

$$L\left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}v\right) = \psi(x)L(v), \quad x \in F, \quad v \in V$$

Corollary 3.5. Let (π, V) be as in ctx.

1. The space of Whittaker functional is precisely 1d.
2. If (π, V) is given in Kirllov form, the space of Whittaker functionals on V are precisely

$$L(\phi) = \lambda\phi(1) \tag{2}$$

   $\lambda \in \mathbb{C}$.

Proof. Step 0. Existence of Kirillov model was proven (but not uniqueness). We suppose (π, V) is in such a form. Eq. 2 is a linear form. We prove there are no others.

Step 1. Twist a general element in V in to the Schwartz space. □

[4] Why do we pass to $\mathcal{S}(F)$?

3.3 Uniqueness of Kirillov model

3.6. (π, V) is as ctx. With its Kirillov model.

Proposition 3.7. Kirillov model of (π, V) is unique. [God70, 5].

Proof. Step 0. Set up. Let (π', V') be a represemtation equivalent to (π, V), where $V' \subset \mathrm{Map}(F^\times, \mathbb{C})$ whose restriction to D_F is ψ_F. Let $A : V' \to V$ denote the iso of G_F-repn.

Step 1. Inducing new Whittaker functional.

Step 1a. Define $L\phi := (A\phi)(1)$ *for* $\phi \in V$. If we show that L is Whittaker functional then $A\phi = \lambda\phi$, for some $\lambda \in \mathbb{C}$. Thus $V = V'$ with $\pi(g) = \pi'(g)$ (using the fact that ϕ is also an iso.)

Step 1b. Checking that L as defined is indeed a Whittaker functional. This is a simple computational check and N_F linearity.

$$L\left(\pi\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\right) = \left(\pi'\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}(A\phi)\right)(1) = \psi(x)L(\phi(1))$$

$\square$

3.4 Uniqueness of Whittaker Model

Definition 3.8. Let $\mathcal{W}(\psi)$ be subspace of $\mathrm{Map}(G_F, \mathbb{C})$ st.

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)W(g)$$

This is a G_F-repn via right regular action, denoted $(\rho, \mathcal{W}(\psi))$, i.e.

$$(\rho(h)W)(g) = W(gh)$$

Theorem 3.9. [JL70, 2.14] Let (π, V) be as in ctx. Then π has a unique Whittaker model.

Proof. Step 0. Existence. We define an injectiono of G_F-modules,

$$V \hookrightarrow \mathrm{Map}(G_F, \mathbb{C}), \quad \phi \mapsto W_\phi$$

$$W_\phi(g) := (\pi(g)\phi)(1) \tag{3}$$

whose image is in $\mathcal{W}(\psi)$. There are a few things to be checked.

1. Well defined, i.e. the image indeed lies in $\mathcal{W}(\psi)$. Now

$$W_\phi(n_x g) = (\pi n_x \pi(g)\phi)(1)) = \psi(x)(\pi(g)\phi)(1)$$

2. The maps is clearly $\mathbb{C}$ -linear. It is G_F-equivariant too:

$$W_{\pi(h)\phi}(g) = (\pi(g)\pi(h)\phi)(1) = W_\phi(gh) = (\rho(h)W_\phi)(g)$$

3. Injectivity. Note

$$W_\phi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right) = \phi(a)$$

so ϕ is zero iff W_ϕ is.

Step 1. Uniqueness. This proof imitates that of 3.7 □

4 Contragredient Representation

4.1 Action of Hecke algebra

[Bum98, p.428], [Vin08, e.2]

4.1. In the tdlc group G_F of interest, the left and right Haar measures conincide: G is *unimodular*. We will assume this for tdlc considered.

Example 4.2. Right and left reg rep. We can define rep'ns ρ, λ of G on $\mathrm{Map}^\infty(G)$ by

$$(\rho(g)f)(h) = f(gh), \quad (\lambda(g)f)(h) = f(g^{-1}h)$$

Definition 4.3. Hecke algebra. We denote

$$\mathcal{H}_G \coloneqq \mathrm{Map}_c^\infty(G, \mathbb{C})$$

We make $\mathcal{H}_G$ an algebra (without unit) under convolution

$$(\phi_1 * \phi_2)(h) \coloneqq \int_G \phi_1(hg^{-1})\phi_2(g)\, dg$$

To show that $\mathcal{H}_G$ is indeed closed under $*$ requires justification, see 4.5.

4.4. There is a rather explicit description of an element $f \in \mathrm{Map}_c^\infty(G, V)$. Let $Y = \mathrm{supp}(f)$.

1. For all $y \in Y$, choose an open nhood U_y of y, such that f is constant.
2. Since Y is cpt by definition,

$$Y = \bigcup_1^m U_i$$

   where we may assume U_i are disjoint. Since each U_i is the complement of finite union of open sets, they are closed and cpt.
3. We obtain that

$$f = \sum_1^m c_i 1_{U_i}$$

4.5. Using the representation in 4.4, let $f_1, f_2 \in \mathrm{Map}_c^\infty(G, V)$. The computation of $f_1 * f_2$ boils down to computing $f_1 = 1_K, f_2 = 1_H$ for two distinct cpt. open subgrps $H, K \subset G$. But we have

$$1_H \star 1_K(s) = \int_K 1_H(sg^{-1})\, dg$$

so that the value in integral is nonzero iff $sg^{-1} \in K$ iff $s \in HK$.

This implies cpt support:

$$\mathrm{supp}\,(1_H * 1_K) \subset HK$$

where the RHS is cpt by [...].

We also have local constancy: consider a open nhood of identity contained in $K \cap H$ [...]

4.6. Action of Hecke algebra. $\mathcal{H}_G$ acts on V by π

$$\pi(\varphi)(v) := \int_G \phi(g)\pi(g)v\,dg \tag{4}$$

4.7. Finite action. Here is where the fact that V is a *smooth* representation comes in handy. For a fixed choice of v, there is an open cpt. K_0 that fixes v. As ϕ is locally constant, we can choose K_0 sufficiently small for which ϕ is constant. [5]

We may replace 4 by

$$\frac{1}{\mathrm{vol}(K_0)}\sum_i \phi(g_i)\pi(g_i)v$$

This shows us that the action of G is purely "algebraic".

4.2 Hecke alg. of cpt. open sbgrp.

4.8. We let K_0 denote an open cpt. subgroup of G. The 0 is to signify "open".

Definition 4.9. Important subalgebra *with* identity. We let

$$\mathcal{H}_{G,K_0} := \{\phi \in \mathcal{H}_G :, \phi(k_1 g k_2) = \phi(g) \text{ for all } k_1, k_2 \in K_0\}$$

denote the K_0 biinvariant functions. We sometimes omit G when context clear.

Corollary 4.10. $\mathcal{H}_{K_0}$ satisfies:

1. It has identity
$$\epsilon_{K_0} := \mu(K_0)^{-1} 1_{K_0}$$
where μ is the Haar measure on G. [6]

2. $\mathcal{H}_{K_0} = \epsilon_{K_0}\mathcal{H}_G\epsilon_{K_0}$, which is $\mathcal{H}_G[\epsilon_{K_0}]$ in lang. of idem. alg.

3. $\mathcal{H}_{K_0}$ is a $\mathbb{C}$-algebrs with unit ϵ_{K_0} and subalgebra of $\mathcal{H}$.

Proof. 2. $\subset$. This is a standard cmptn. If ϕ is right invariant on action of K_0,

$$\phi * 1_{K_0}(h) = \frac{1}{\mu(K_0)}\int_K \phi(hg^{-1})\,dg = \phi(h) \tag{5}$$

$\square$

Proposition 4.11. [Vin08, e.2.2]

$$\mathcal{H}_G = \bigcup_{K_0 \subset_{\text{cpt,open}} G} \mathcal{H}_{K_0}$$

[5] Precisely, the argument goes as follows: we let W be an open set containing e for which ϕ is constant. Since G is tot. disc. there is a compact open $W' \subset W$. Define our new cpt. open to be $W' \cap K_0$.

[6] Note that in general for any compact open set S, $\epsilon_S := \mu(S)^{-1}1_S \in \mathcal{H}_G$.

4.3 Idempotented algebras

[Bum98, 3.4].

4.12. In the context of idempotented algebras, the ring can be *non*-comm., and *non*-unit.

4.13. In this section we will discuss

- Definition of idempotent algebras.
- Modules over idempotented algebras.

Definition 4.14. Basic notions in idempotent.

- An element $e \in R$ is *idempotent* if $e^2 = e$.
- The idpt. forms a poset by defining
$$e \leq f \Leftrightarrow ef = fe = e$$

Example 4.15. Com. model. Let X be a set, then the set of $\mathbb{C}$-valued functions, $\mathrm{Map}(X, \mathbb{C})$, form a com. $\mathbb{C}$-alg via pt. wise add. and mult. The characteristic functions are idempotents. Also

$$\chi_B \leq \chi_{B'} \Leftrightarrow B \subset B'$$

4.16. A non. com. model is the Hecke algebra with convolution, $\mathcal{H}_G$.

Definition 4.17. An *idempotented algebra* over k, is a k-algebra H with a collection of idem. E st.

1. For all $e_1, e_2 \in E$, exists $f \in E$, such that $e_1, e_2 \leq f$.
2. For all $x \in H$, there exists $e \in E$, with $ex = xe = x$.

Following convention when we speak of an idemp. in (H, E) we mean an element in E.

Definition 4.18. Subrings induced from idpts. If H is a ring, $e \in \mathrm{Idem}(H)$, then write

$$H[e] = eHe$$

Definition 4.19. Submodules induced from idpt. alg.

Corollary 4.20. If $e \leq f$ then $H[e]$ is a subring of $H[f]$.

Proof. Choose an arbitrary element $ebe \in H[e]$, then $ebe = febef \in H[f]$. ☐

4.21. Now we discuss modules over idempotented algebras. Let H be an idpt. alg, M a H-module.

- M is *smooth* if
$$\mathrm{colim}_e\, M[e] \to M \tag{6}$$
is an iso.
- M is *admissible* if eM is fd over k for all $e \in E$.

4.22. Smoothness is equivalent to the condition if $x \in M$, then exists $e \in E$, such that $ex = x$.

Proof. $\Rightarrow$. iso. of 6 implies every element of $x = M$ is of the form yx', for some $y \in E, x' \in M$. Hence $yx = x$. $\Leftarrow$. 6 in general is an inj. Hypothesis yields surj. ☐

Theorem 4.23. [JL70, p25] Let G be tdlc. We regard $\mathcal{H}_G$ as an idempotented alg[7]

1. Let (π, V) be a smooth. repn. of G. If $f \in \mathcal{H}_G$, then we obtain a smooth repn of $\mathcal{H}_G$ of V by

$$\pi(f)v = \int_G f(g)\pi(g)v\,dg \tag{7}$$

   This induces smooth. repn $\mathcal{H}_G$ on V.

2. The construction is
   - a bij. on smooth repn.
   - a bij. on irr. repn.
   - stable subspaces.

Proof. 1. *Step 0. Making sure the map is well defined.* The function $g \mapsto f(g)\pi(g)v$ is in $\mathrm{Map}_c^\infty(G, V)$. Hence, the sum is finite. Change of var. shows

$$\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)v$$

Step 0a. This is a smooth representation. Nts. exists $\xi \in \mathrm{Idem}(\mathcal{H}_G)$

$$\pi\xi v = v$$

Step 2. Constructing inverse map. For $v \in V$, write [8]

$$v = \sum_1^n \pi f_i v_i$$

$$\pi(g) \cdot v = \sum_1^n \pi(\lambda(g) \cdot f_i)v_i$$

λ where λ is the left reg. action of G on $\mathcal{H}_G$. □

4.24. Now let us discuss the contragredient representation. We will suppose our idempotented algebra has an anti involution, ι. e.g. in $\mathcal{H}_G$, $f^\iota(x) := f(x^t)$.

Definition 4.25. Let M ve a H-module.

4.4 Contragredient representation

4.26. Let (π, V) be an adm. repn of G_F.

4.27. The *contragredient repn.*

- If $\hat{v} : V \to \mathbb{C}$ is a linear functional, we write, for $v \in V$,

$$\langle v, \hat{v} \rangle := \hat{v}(v)$$

  $\hat{v}$ is *smooth* if there exists an open nhood U of identity in G such that

$$\langle \pi(g)v, \hat{v} \rangle = \langle v, \hat{v} \rangle$$

  for all $g \in U$, $v \in V$. We denote by $\hat{V}$ the set of all smooth functions on V.

[7] with 1_U as idem. where U is cpt open.
[8] Note that we can take $i = 1$ by definition of smoothness.

- We have the *contragradient represention* $(\hat{\pi}, \hat{V})$, given by

$$\langle v, \hat{\pi}(g)\hat{v}\rangle = \langle \pi(g^{-1})v, \hat{v}\rangle$$

Remark 4.28. There are two points to be made.

1. One has to check $\hat{\pi}(g)\hat{v}$ is smooth. Indeed, we wish to find an open sbgrp. W such that for all $v \in V$,

$$g \cdot \hat{v}(v) = g \cdot \hat{v}(Wv)$$

where we wrote g for $\hat{\pi}g$. This can be restated as

$$\hat{v}(g^{-1}v) = \hat{v}(\left(g^{-1}Wg\right) g^{-1}v)$$

for all $v \in V$. Thus, our desired open set is $W = gUg^{-1}$, where U is given from definition 1. of 4.27.

2. The very definition of smooth functions guarantees that $(\hat{\pi}, \hat{V})$ is smooth rep'n.

4.29. In the pursuing discussion (π, V) is inf. dim. and irred. We denote its central quasi-char. by ω.

Lemma 4.30. [JL70, p73] Let (π, V) be taken in Kirllov form. The space of Kirllov model of $\omega \otimes \pi$ consists of linear functionals $\omega\phi$, with $\phi \in V$.

Proof. Let (π', V') be the subrepn of (π, V) consissting of $\omega\phi$ with $\phi \in V$. We can equivalence map

$$(w \otimes \pi, V) \to (\pi', V'), \quad \phi \mapsto \omega\phi$$

it suffices to check the action on V' restricts to ξ_ψ. □

Theorem 4.31. [JL70, 2.18]

$$\hat{\pi} \simeq w^{-1} \otimes \pi$$

Proof. Step 0. Defining a bilinear form. We begin by proposing a possible bilinear map

$$\langle \phi, \phi' \rangle = \int_{F^\times} \phi(a)\phi'(-a)d^\times a \tag{8}$$

□

5 Appendix

5.1 Topological Groups

5.1. We recall some topological notions. We let $X \in \mathrm{Top}$, G a topological group.

Basics.

- A space X is *hom.* if given any two points $x, y \in X$, exists $f : X \to X$ such that $fx = y$.

Local compactness and connectedness.

- X is *locally cpt.* if for all $x \in X$.
- G is a *loc. cpt. grp* if it is Hausdorf and loc. cpt space.

Example 5.2. Let R be a top. ring. $\mathrm{GL}_n(R)$, $M_n(R)$ are both, top. ring, given the subspace topology in R^{n^2}.

5.3. Now let us list a whole host of properties for a topological group. G

1. If $U \subset G$, then U is open iff tU is open iff Ut is open iff U^{-1} for all $t \in G$.
2. Every nhoodo U of 1 contains an open symmetric nhood V of 1 such that $VV \subset U$.
3. Every open subgroup is also closed.
4.

Proof. 3. Let $H \subset G$ be open subgroup. G can be written as the union of cosets of H. We have the relation

$$Y = \bigcup_{x \in G \setminus H} xH$$

$$H = G \setminus Y$$

□

Proposition 5.4. [Vin08, a.4.1] Let G be a Hausdorff top. grp. Any subgroup of G which is loc. cpt. is closed.

Corollary 5.5. [Vin08, e.4.2] A Hausdorff top. grp. G is loc. cpt. and t.d. iff every nhood of 1 contains a compact open subgroup.

Remark 5.6. Importantly, for those reading the text [BZ76], these are the *l-groups*.

5.2 Smooth and admissible representations

Definition 5.7. Let G be tdlc, (π, V) a representation. V admits *no* topology.

- π is *smooth* if for any $v \in V$, stabilizer [9]

  $$\mathrm{Stab}(v) := \{g \in G : gv = v\}$$

  is an open subgrp of G. This is nonempty as e lies in the grp.

[9] This is a rather abuse of notation, but the context should make it clear.

- If π is smooth, and if for any open subgroup $U \subset G$

$$V^U = \{v \in V : gv = v \text{ for all } g \in U\} \tag{9}$$

  is fin. dim, then π is *admissible*.

5.8. Continuity. I find it more natural to interpret smooth representations as *continuous* representations. By definition, if V is given the discrete topology, then (π, V) is smooth iff it is continuous.

Proposition 5.9. Finite dimensionality. Let (π, V) be a fd. rep. of a tdlc group G Then the following are equivalent.

1. π is admissible.
2. π is smooth.
3. Kernel of π is an open subgroup.
4. π, as a map $G \to \mathrm{GL}(V)$ is continuous.

Proof. $1 \Leftrightarrow 2$ is clear from defn. $2 \Leftrightarrow 3$. Suppose $\ker \pi$ is open. Then for any $g \in \mathrm{Stab}(v)$, $g \ker \pi \subset \mathrm{Stab}(v)$ is an open hood of g. So $\mathrm{Stab}(v)$ is open. Suppose $\mathrm{Stab}(v)$ is open. Let $\{v_i\}$ be a $\mathbb{C}$-basisc of V, so

$$\ker \pi = \bigcap_1^n \mathrm{Stab}(v_i)$$

is open.

$3 \Leftrightarrow 4$.

□

5.10. Irreducible rep'ns.

- If (π, V) is a smooth or admissible rep'n, then every G-invariant subspace of V is also smooth or admissible rep'n respectively.
- A smooth representation (π, V) of G is *irreducible* if V contains no nontrivial G-invariant subspaces.

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